The physics of putting

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Abstract: The motion of a rolling golf ball on a sloped golf green is modeled. The resulting calculated path of a golf ball is then used, along with a model of the capture of the golf ball by the hole, to determine the resulting launch conditions required for a successful putt. Estimates of the probability of making certain putts are also presented.

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Résumé : Nous modélisons le mouvement d'une balle de golf sur un vert en pente. La trajectoire calculée et un modèle décrivant la capture par le trou sont alors utilisés pour déterminer les conditions initiales qui font que le coup roulé est couronné de succès. Nous présentons également les probabilités de succès de certains coups roulés.

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1. Introduction

Whether it be first-time golfers at a pitch and putt or professional golfers playing a regulation 18-hole course, putting is the most common stroke in the game of golf. In professional tournaments a crucial putt is typically the most dramatic shot and the saying, "Drive for show, putt for dough", has been validated many a time. The goal of this paper is to look at some of the physics of putting and to determine the launch conditions required for a successful putt.

The first aspect of putting that will be considered will be the motion of a rolling golf ball on a golf green. Models of the motion of a golf ball on a sloped green have been presented by Lorensen and Yamrom [1] and Alessandrini [2]. These models, however, treat the motion of the golf ball as that of an object sliding along a sloped surface and do not take into account the rolling motion. The model that will be presented in this paper will assume that the ball is rolling as it moves over the green. The second aspect of putting that will be considered is the capture of a rolling golf ball by the hole. Holmes [3] presented a detailed model of the capture of a golf ball by a hole on a flat green. This model will be discussed briefly along with a correction that will be required to account for sloped greens. The models of the motion and path of a golf ball on the green and the capture of the golf ball by the hole will then be used to determine the launch conditions required for successful putts on various sloped greens.

2. The path of a putt

The initial motion of a putted golf ball on a golf green is, in general, more involved than first appearances would indicate. The basic act of putting involves striking a golf ball with a flat faced golf club, the putter. As most putters have a slight loft, a typical putted golf ball will become slightly airborne after being struck and will initially bounce several times on the green. This is especially evident when

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putting on dewy greens where the bounce marks become visible. The frictional force between the green and the golf ball will normally put the ball in a state of pure rolling after the bounce phase. If the putter has no loft, the golf ball will initially be in a combined state of sliding and rolling before it finally ends up in a state of pure rolling. Both Cochran and Stobbs [4] and Daish [5] indicate that a putted golf ball will be in a state of pure rolling after traveling approximately 20% of the total length of the putt. However, this would, in general, depend on both the loft of the putter and on the nature of the impact as a golf ball can initially be given top spin or bottom spin depending on the relative position of the putter at impact. For the purposes of this paper the ball will be taken to be in a state of pure rolling immediately after it leaves the face of the putter. This will greatly simplify the analysis and this approximation would be expected to have only a secondary effect on the actual path of the putt.

2.1. Putting on a level green

The dynamics of a ball rolling along level and inclined surfaces has been modeled by Domenech et al. [6] and Witters and Duymelinck [7]. In general, both the ball and the surface on which it rolls become slightly deformed and this is the source of the retarding force that acts on the ball. The force due to the deformed surface will be distributed over the contact area and, in general, will be equivalent to a single force and a couple acting on the ball. This force and couple are, in turn, equivalent to a single force acting at the point on the ball's surface where the resulting moment is equal to that of the couple. This equivalent force can be resolved into a component, n, normal to the surface and a component, f, tangential to the surface. The position of the equivalent single contact point on the golf ball is given by ρ, the perpendicular distance between the normal component of the contact force and the centre of mass of the golf ball. The force diagram in the case of a golf ball of mass m and radius R, rolling on a level green, is shown in Fig. 1. The resulting equations of motion for a golf ball, with a moment of inertia I, rolling on a level green will then be

$$ma_y = -f \tag{1}$$

$$I\alpha_x = n\rho - fR_\mathrm{t} \tag{2}$$

where R_t, the perpendicular distance between the tangential component of the contact force, f, and the centre of mass of the golf ball, is given by

$$R_\mathrm{t} = (R^2 - \rho^2)^{1/2} \tag{3}$$

Typically $\rho << R$ and the approximation that $R_\mathrm{t} = R$ will be used in the analysis. The constraint of rolling will be given by

$$a_y = -\alpha_x R \tag{4}$$

Solving (1), (2), and (4) for the acceleration of the golf ball results in,

$$a_y = -\frac{5}{7}\rho_\mathrm{g} g \tag{5}$$

where $\rho_\mathrm{g} = (\rho/R)$. The above model ignores the fact that the surface of a golf ball is dimpled, however, as the dimpled surface would be expected to have only a minor effect on the path it seems a reasonable approximation to treat its surface as smooth.

Experimental measurements of a golf ball rolling on a green by Hubbard and Alaways [8] have indicated that there is a dependence of the deceleration of a golf ball on its speed, with the retarding force increasing at lower speeds. However, the dependence was found to be small, i.e., a 10% variation over a 14 ft (4.3 m) putt (1 ft = 0.3048 m), and for the purposes of this paper the golf ball's deceleration, and therefore the value of ρ_g, will be taken to be constant. In the case of a relatively hard golf ball rolling

Fig. 1. The forces acting on a golf ball rolling on a level green.

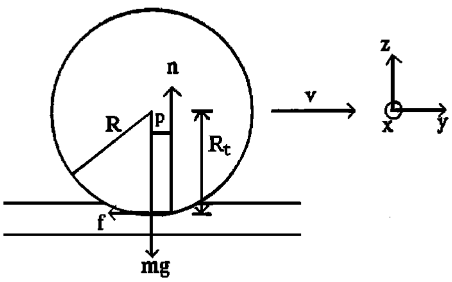
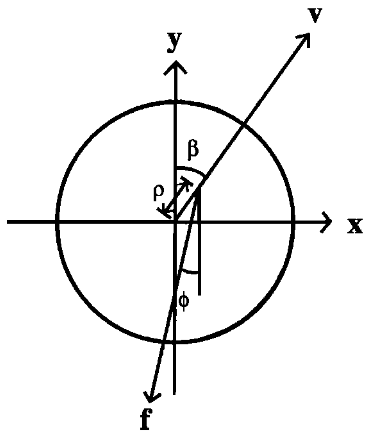


Fig. 2. The overhead view for a golf ball launched at a speed v and a launch angle β.



on a compliant green, the value of ρ_g would be expected to be primarily determined by the firmness of the green and the condition of the grass surface. In golf, one refers to the speed of the green, with a fast green being one where the ball rolls a relatively long distance before coming to rest. The speed of a green will be directly related to the deceleration of the golf ball and will, therefore, be a measure of the value of ρ_g. The speed of a green is typically measured by a device called a stimpmeter, which is basically an inclined plane with a V-groove running down its centre. Holmes [9] has shown that the initial speed of a golf ball when it leaves the end of a stimpmeter is 1.83 m/s. For what would be considered a very fast green the ball rolls, after leaving the end of the stimpmeter, a distance of approximately 12 ft (3.66 m). For what would be considered a very slow green the ball rolls a distance of only approximately 4 ft (1.22 m). Using the speed of the golf ball as it leaves the stimpmeter (as determined by Holmes), the above extreme roll distances, and the acceleration of the golf ball as given by (5), the range of values for ρ_g with golf greens can be found. The result is that for golf balls rolling on golf greens

$$0.065 < \rho_g < 0.196 \tag{6}$$

with an average value of 0.131.

2.2. Putts on sloped greens

For the more general case of a rolling golf ball on a sloped green, the value of ρ_g will be taken to be the same as is found with level greens, and the equivalent contact point on the golf ball will be taken to be along the direction of travel. These approximations will greatly simplify the analysis and would be expected to have only a secondary effect on the determined paths.

Figure 2 shows the overhead view for a golf ball launched at a speed of v and a launch angle of β towards a hole that lies on the y-axis. The normal component, $\boldsymbol{n}$, of the contact force acting on the golf ball will be given by

$$\boldsymbol{n} = n\boldsymbol{k} \tag{7}$$

while the tangential component, $\boldsymbol{f}$, of the contact force is given by

$$\boldsymbol{f} = -f(\sin\phi\boldsymbol{i} + \cos\phi\boldsymbol{j}) \tag{8}$$

where the angle ϕ is as shown in the figure. Taking the surface of the green to be sloped at angles, with respect to the horizontal, of θ along the x-axis and φ along the y-axis, the gravitational force will be given by

$$\boldsymbol{W} = -mg(\sin\theta \boldsymbol{i} + \cos\theta \sin\varphi \boldsymbol{j} + \cos\theta \cos\varphi \boldsymbol{k}) \tag{9}$$

Using Newton's second law

$$m\boldsymbol{a} = \boldsymbol{W} + \boldsymbol{f} + \boldsymbol{n} \tag{10}$$

then results in the following three equations:

$$ma_x = -mg \sin\theta - f \sin\phi \tag{11a}$$
$$ma_y = -mg \cos\theta \sin\varphi - f \cos\phi \tag{11b}$$

and

$$0 = n - mg \cos\theta \cos\varphi \tag{11c}$$

The position of the contact point with respect to the centre of mass of the golf ball, in the case where $\rho << R$, will be

$$\boldsymbol{r} = \rho \sin\beta \boldsymbol{i} + \rho \cos\beta \boldsymbol{j} - R\boldsymbol{k} \tag{12}$$

It then follows from

$$I\boldsymbol{\alpha} = \boldsymbol{r} \times (\boldsymbol{n} + \boldsymbol{f}) \tag{13}$$

that

$$I\alpha_x = n\rho \cos\beta - fR \cos\phi \tag{14a}$$

and

$$I\alpha_y = -n\rho \sin\beta + fR \sin\phi \tag{14b}$$

The constraint of rolling for this general two-dimensional case will be

$$\boldsymbol{a} = \boldsymbol{\alpha} \times \boldsymbol{R} \tag{15}$$

from which

$$a_x = \alpha_y R \tag{16a}$$

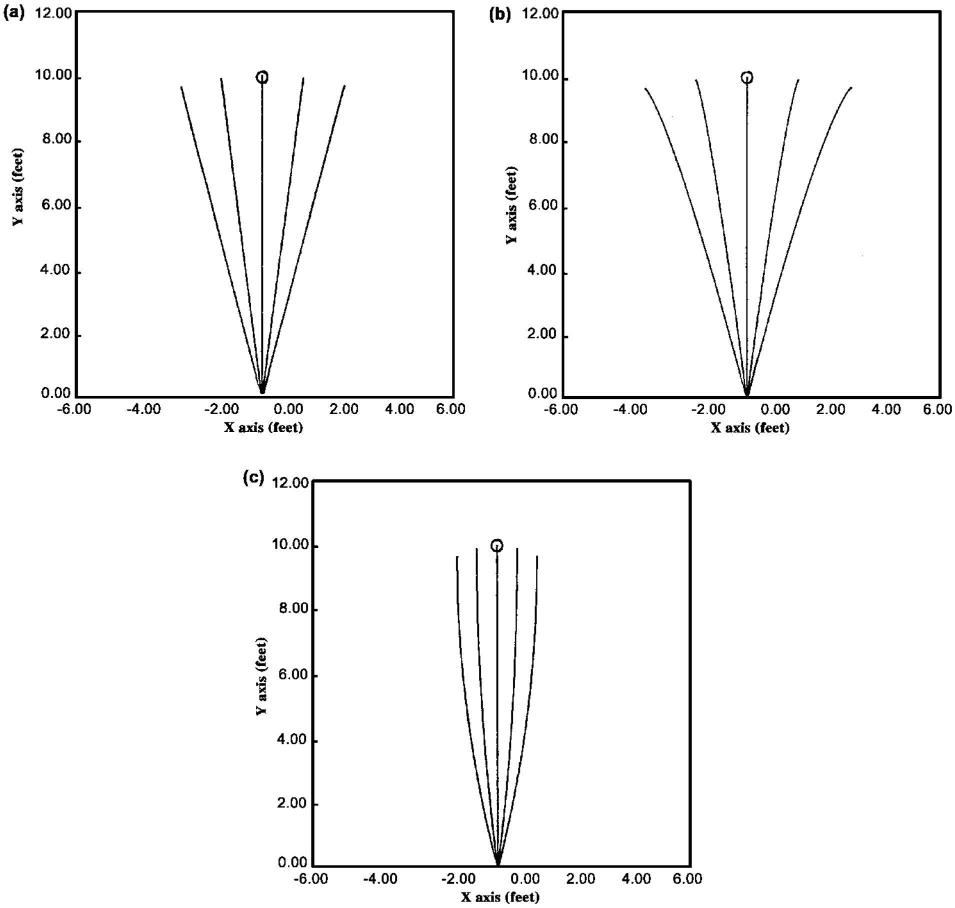
and

$$a_y = -\alpha_x R \tag{16b}$$

Solving (11), (14), and (16) for the direction and magnitude of the tangential component of the contact force, results in

$$\tan\phi = \frac{\rho_{\mathrm{g}} \cos\theta \cos\varphi \sin\beta - I_{\mathrm{b}} \sin\theta}{\rho_{\mathrm{g}} \cos\theta \cos\varphi \cos\beta - I_{\mathrm{b}} \cos\theta \sin\varphi} \tag{17a}$$

Fig. 3. The paths of golf balls launched at angles of $-15°$, $-7.5°$, $0°$, $7.5°$, and $15°$ along (*a*) a level green, (*b*) a green with an uphill slope of $5°$, and (*c*) a green with a downhill slope of $5°$.



and

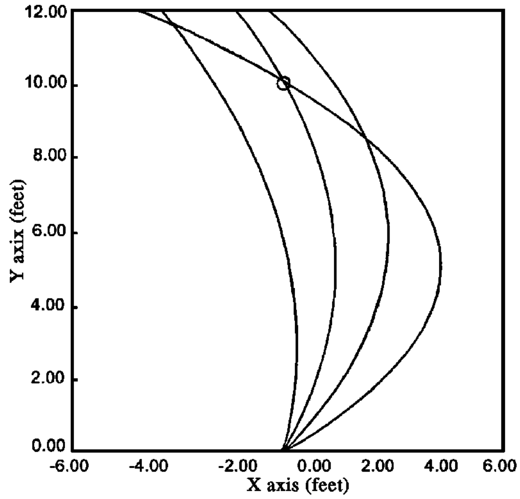
$$f = \frac{\rho_{\mathrm{g}} \cos\theta \cos\varphi \cos\beta - I_{\mathrm{b}} \cos\theta \sin\varphi}{(1 + I_{\mathrm{b}}) \cos\phi} mg \qquad (17b)$$

where $I_{\mathrm{b}} = I/mR^2$. The golf ball will be modeled as a uniform solid sphere with I_{b} equal to 2/5. The above expressions for f and ϕ along with (11a) and (11b) will allow the x- and y-components of the acceleration of the golf ball to be determined for greens of various slopes. Given these accelerations, along with the initial launch conditions, the paths of the putted golf balls can be determined. For the paths shown in this paper the step size used in the calculations was 0.001 s, which resulted in a calculation uncertainty of less than 0.1% in the determined paths. Two specific cases of the paths of putts on sloped greens will be considered in this paper. These are putts to holes that are either directly uphill or downhill from the initial position of the golf ball and putts that are to holes that are directly across the slope of a green, i.e., slopes that run along the x-axis of Fig. 2.

2.3. Uphill and downhill putts

In the case of straight uphill and downhill putts, θ is equal to zero in (11) and (17). As an example of calculated paths, Fig. 3 shows the determined paths of golf putts for a range of launch angles in the

Fig. 4. The paths of golf balls launched at angles of 5°, 30°, 45°, and 60°, on a green with a 5° slope along the x-axis and with $\rho_g = 0$.



case of putts towards a hole 10 ft away along a flat green, a green with an uphill slope of 5°, and a green with a downhill slope of 5°. A value of 0.131 was used for ρ_g in all three cases and the launch velocities of 2.37, 3.05, and 1.36 m/s of the golf balls, for each of the respective slopes, were selected so that a putt directly along the y-axis would travel exactly 10 ft. As can be seen, in the case of uphill putts, putts that are initially off line will diverge from the target, while in the case of downhill putts, they will converge towards the target. In general, it was also found that the amount of divergence for uphill putts and the amount of convergence for downhill putts increases for faster greens, i.e., smaller values of ρ_g. However, in the case of downhill putts, if the value of ρ_g is too small, the ball will have a positive acceleration along the y-axis and will, therefore, not come to rest if the hole is missed. The critical value for ρ_g can be determined by substituting $\phi = 0°$ and $\beta = 0°$, along with $\theta = 0°$ into (17b) and then substituting for f in (11b). The result is that a_y will be positive for a downhill putt if

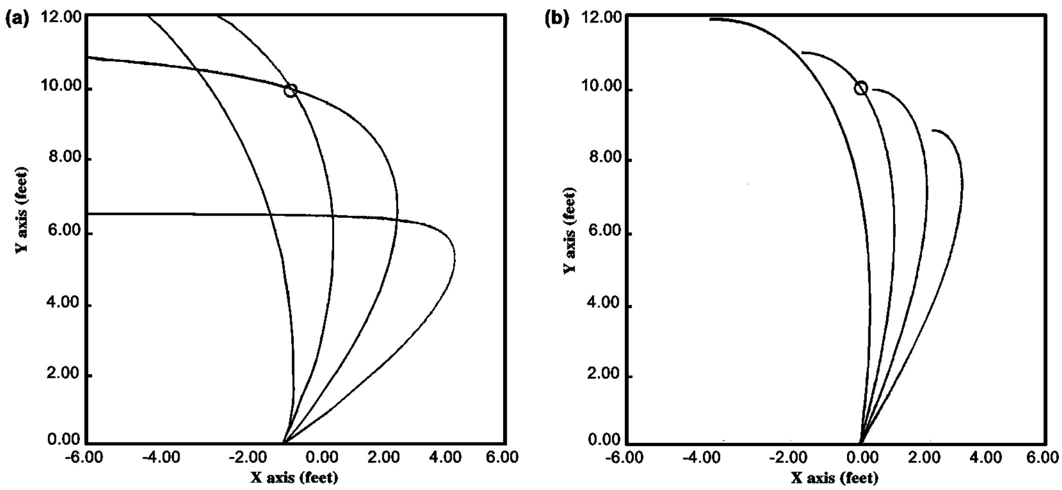
$$\rho_g < \tan \mid \varphi \mid \tag{18}$$

and, therefore, for a downhill slope of 5° the critical value for ρ_g will be 0.087. For values of ρ_g less than this critical value the golf ball will accelerate downhill and will not come to rest if it misses the hole.

2.4. Putts across the slope

In the case of putting across the slope of a green, φ will be equal to zero in (11) and (17). The resulting paths for putts with various launch conditions were determined for various values of ρ_g for the case of a slope of 5° along the x-axis. Figure 4 shows the case where there is no retarding force, $\rho_g = 0$. The launch speed of the golf ball was set at 1.74 m/s and the paths that the golf balls would follow for various launch angles towards a hole 10 ft away are shown. From (11), in the case of no retarding force, $f = 0$, a golf ball will have a constant acceleration of $-g \sin\theta$ along the x-axis and will have no acceleration along the y-axis. As is seen in Fig. 4, the golf ball will, therefore, follow a parabolic path and, as is the case with projectiles, for any two launch angles that are complementary, golf balls that have the same launch speed will travel the same range, defined as the distance traveled outwards up to the point that the ball returns back to $x = 0$. Figure 5 shows similar examples of paths of putts across a 5° sloped green for values of ρ_g of 0.065 and 0.131, corresponding to a fast and an average green. The launch speeds in the two cases are 2.00 and 2.50 m/s, respectively. As is seen in

Fig. 5. (*a*) The paths of golf balls launched at angles of 5°, 22°, 39°, and 56° on a green with a 5° slope along the x-axis and with $\rho_g = 0.065$.(*b*) The paths of golf balls launched at angles of 8°, 16°, 24°, and 32° on a green with a 5° slope along the x-axis and with $\rho_g = 0.131$.



Figs. 5*a* and 5*b*, the same general result, as was found in the case of no retarding force, holds except that the same range is obtained for launch angles summing to approximately 60° and 40°, respectively, for the values of ρ_g considered. Also, as is seen in Fig. 5*b*, for larger values of ρ_g, which translates into greater retarding forces, there may be only one launch angle, for a given launch speed, for which the golf ball will actually reach the hole.

The model of the path of rolling golf balls on sloped greens that has been presented has provided reasonable results. However, it must be made clear that the model can only approximate the actual behavior of a real putt. This is not only due to the approximations made in the treatment of the contact force and the initial motion of the golf ball but also because the grass surface will have small but numerous imperfections that will result in deviations in the golf ball's path.

3. The capture of a putt

The problem of the capture of a golf ball by a hole, of radius R_H, on a flat green is considered in the paper by Holmes [3]. The simplest condition for the capture of a golf ball that is traveling directly towards the centre of the hole is for the golf ball, after it leaves the front rim, to free fall a distance greater than its radius, R, before it strikes the far rim. The critical case is shown in Fig. 6. The time of flight of the free-falling golf ball will be given by

$$t = \frac{(2R_H - R)}{v_f} \tag{19}$$

where v_f is the speed of the golf ball when it reaches the hole. The condition for capture by free fall will, therefore, be

$$\frac{gt^2}{2} > R \tag{20}$$

or in terms of v_f

$$v_f < (2R_H - R)\left(\frac{g}{2R}\right)^{1/2} \tag{21a}$$

Fig. 6. The free-fall capture of a golf ball of radius R by a hole of radius R_H on a level green.

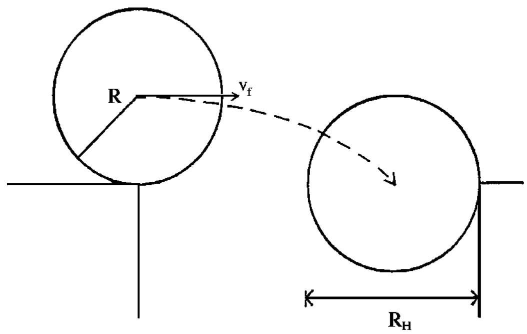
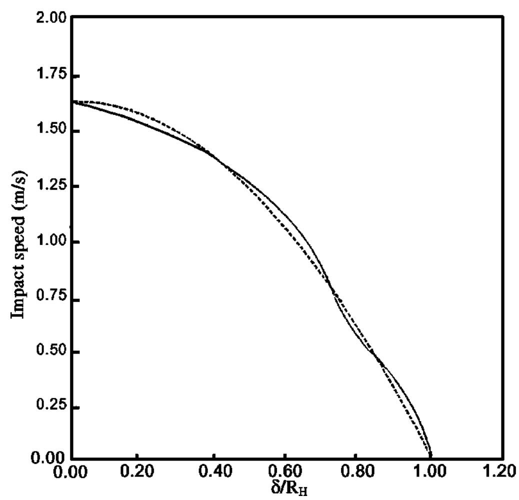


Fig. 7. The impact-speed – impact-parameter capture space for a golf ball impacting on a golf hole as determined by Holmes (continuous line) and as approximated by (23) (broken line).



For a standard USGA-approved golf ball of radius 2.135 cm and a hole of radius 5.40 cm this becomes

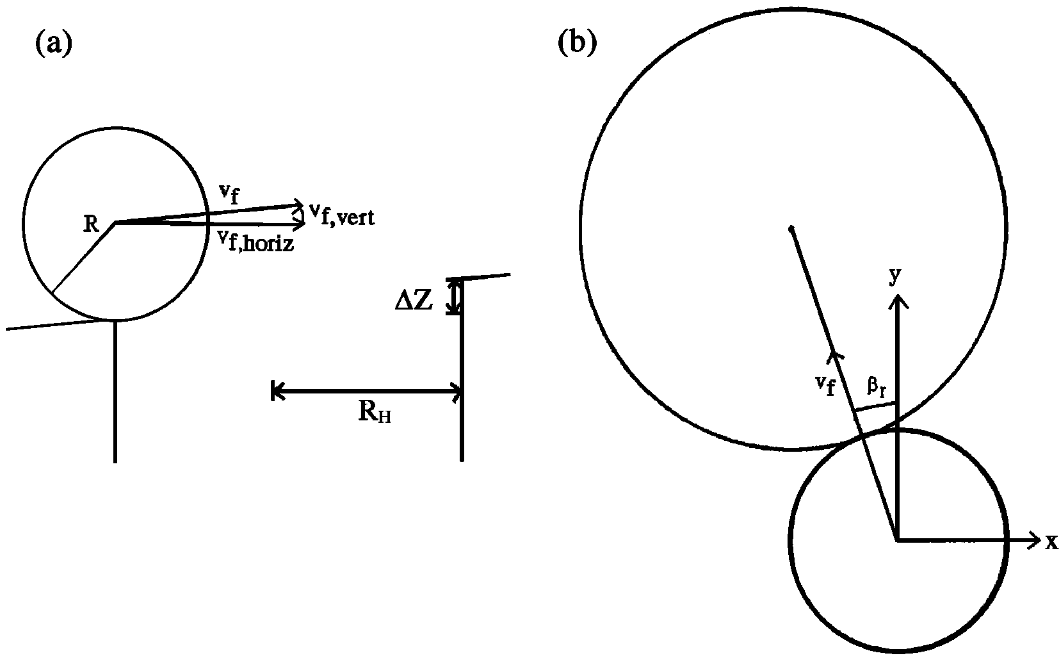
$$v_f < 1.31 \ \ \mathrm{m/s} \tag{21b}$$

In the more general case of off-centre impacts the condition for capture by free fall is shown by Holmes to be given by

$$v_f < \left((R_H^2 - \delta^2)^{1/2} + \left((R_H - R)^2 - \delta^2\right)^{1/2}\right)\left(\frac{g}{2R}\right)^{1/2} \tag{22}$$

where δ, the impact parameter, must be less than $R_H - R$, if the golf ball is to leave the near edge before it strikes the far edge. If the golf ball's speed and impact parameter does not meet the above condition it may still be captured as the ball may roll on the lip of the hole before falling in or it may strike the far side of the hole after falling a distance less than R, but bounce and then fall in the hole. Holmes considered all the possible ways that a golf ball may be captured by a hole and by computer modeling determined the impact-speed – impact-parameter capture space for a golf ball impacting on a golf hole. The result is shown in Fig. 7. As is seen, in the case of a direct impact, $\delta = 0$, the critical capture speed is increased from 1.31 m/s for capture by free fall to 1.63 m/s when all methods of capture are

Fig. 8. The free-fall capture of a golf ball of radius R by a hole of radius R_H on an uphill sloped green; (*a*) side view and (*b*) overhead view.



considered. Holmes' model matched well with experimental measurements, which he also carried out. The following function, which is also shown in Fig. 7, provides a reasonable fit to the boundary of the capture region in Holmes' model and will be used in the analysis:

$$v_c(\delta) = 1.63\ \text{m/s} - (1.63\ \text{m/s})\left(\frac{\delta}{R_H}\right)^2 \tag{23}$$

To take into account putting on a sloped green, a correction to the capture region determined by Holmes is required. The case of an uphill putt on a green of slope φ in the case of direct impact, $\delta = 0$, and an approach direction of β_f is shown in Fig. 8. The time of flight in this case will be given by

$$t = \frac{(2R_H - R)}{v_{f,\,\text{horiz}}} \tag{24}$$

where $v_{f,\,\text{horiz}}$, the horizontal velocity component of the golf ball, is given by

$$v_{f,\,\text{horiz}} = v_f(\cos^2\beta_f\cos^2\varphi + \sin^2\beta_f)^{1/2} \tag{25}$$

The condition for capture by free fall will then be given by

$$\frac{gt^2}{2} - v_{f,\,\text{vert}}t > R - \Delta z \tag{26}$$

where $v_{f,\,\text{vert}}$, the vertical velocity component of the golf ball, is given by

$$v_{f,\,\text{vert}} = v_f\cos\beta_f\sin\varphi \tag{27a}$$

and Δz, the vertical distance between the near and far sides of the hole, is given by

$$\Delta z = 2R_H\cos\beta_f\sin\varphi \tag{27b}$$

For relatively small slopes, $v_{\mathrm{f,\,horiz}} = v_{\mathrm{f}}$, and the condition for capture by free fall, (26), will then become

$$v_{\mathrm{f}} < (1 - \cos\beta_{\mathrm{f}} \sin\varphi)^{-1/2}(2R_{\mathrm{H}} - R)\left(\frac{g}{2R}\right)^{1/2} \tag{28}$$

This expression also holds in the case of downhill putts with φ being negative. In the case of putts across the slope of a green the condition for capture by free fall is found by similar analysis to be

$$v_{\mathrm{f}} < (1 + \sin\beta_{\mathrm{f}} \sin\theta)^{-1/2}(2R_{\mathrm{H}} - R)\left(\frac{g}{2R}\right)^{1/2} \tag{29}$$

Although (28) and (29) only apply to capture by free fall in the case of direct impact, $\delta = 0$, as an approximation of the effect slope has on the capture of a golf ball, these corrections for a sloped green will be applied in general to Holmes' model. This approximation would only be reasonable in the case of the relatively small slopes that are considered in this paper. Therefore, a golf ball will be taken to be captured by the hole, in the case of uphill or downhill putts on a green of slope φ, if its impact speed v_{f}, impact parameter δ, and approach direction β_{f} satisfy

$$v_{\mathrm{f}} < (1 - \cos\beta_{\mathrm{f}} \sin\varphi)^{-1/2} v_{\mathrm{c}}(\delta) \tag{30}$$

In the case of uphill putts on a green with a slope of 5° the result is that the critical capture speed for direct impacts increases from 1.63 to 1.71 m/s and in the case of downhill putts on the 5° sloped green it decreases to 1.56 m/s. Similarly, a golf ball will be taken to be captured by the hole for putts across the slope of a green, if its impact speed v_{f}, impact parameter δ, and approach direction β_{f} satisfy

$$v_{\mathrm{f}} < (1 + \sin\beta_{\mathrm{f}} \sin\theta)^{-1/2} v_{\mathrm{c}}(\delta) \tag{31}$$

4. Required launch conditions

Given the above model for the motion and path of a golf ball, along with the slope-adjusted capture model of Holmes, the required launch conditions of a golf ball for a successful putt can be determined. The cases of putting on a level surface, putting on uphill and downhill slopes, and putting across the slope of a green will be considered.

4.1. Putting on a level surface

In the case of putting on a level surface, with a constant acceleration as given by (5), the speed of the golf ball, v_{f}, when it reaches the hole, located a distance y away, will be given by

$$v_{\mathrm{f}} = \left(v_{\mathrm{o}}^2 - \left(\frac{10}{7}\right)\rho_{\mathrm{g}} g y\right)^{1/2} \tag{32}$$

where v_{o} is the launch speed of the golf ball. Therefore, in the case of direct impacts, where the range of impact speeds that result in the capture of the golf ball on a level green is given by

$$0 < v_{\mathrm{f}} < 1.63\ \ \mathrm{m/s} \tag{33}$$

the required launch speeds for a successful putt will be given by

$$\left(\left(\frac{10}{7}\right)\rho_{\mathrm{g}} g y\right)^{1/2} < v_{\mathrm{o}} < \left((1.63\ \ \mathrm{m/s})^2 + \left(\frac{10}{7}\right)\rho_{\mathrm{g}} g y\right)^{1/2} \tag{34}$$

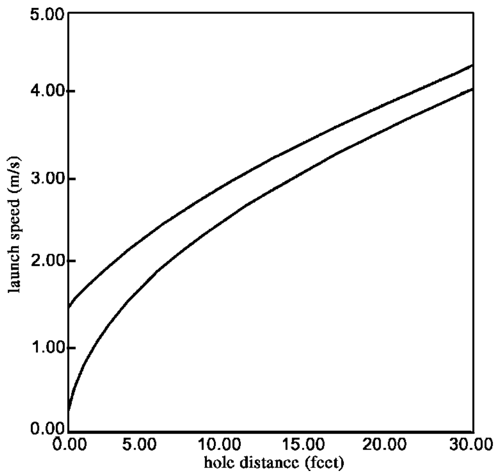


Fig. 9. The range of possible launch speeds for a successful putt as a function of hole distance in the case of a level green with $\rho_g = 0.131$.

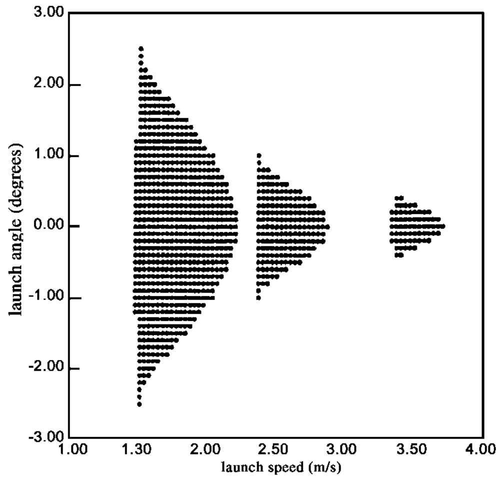


Fig. 10. The launch conditions that resulted in successful putts for hole distances of 4 ft (left), 10 ft (middle), and 20 ft (right) for a level green with $\rho_g = 0.131$.

Figure 9 shows the range of possible launch speeds for a successful putt as a function of the hole distance in the case of a green with $\rho_g = 0.131$. As is shown, the range of allowed launch speed decreases with hole distance. The range of possible launch angles that can lead to a successful putt to a hole a distance y away on a level green will in turn be given by

$$-\tan^{-1}\left(\frac{R_H}{y}\right) < \theta < \tan^{-1}\left(\frac{R_H}{y}\right) \tag{35}$$

To determine the complete set of launch speeds and launch angles that would lead to a successful putt, the paths that golf balls would travel, given various launch conditions, were determined. For those putts whose paths crossed the hole, the capture condition as given by (30), with $\varphi = 0$, was checked. The resulting launch conditions that resulted in successful putts to hole distances of 4, 10, and 20 ft for a level green with $\rho_g = 0.131$ are shown in Fig. 10. In the case where the probability of a player making a putt is small, the scatter in the launch speed and launch angles in the putts of the given player will be much larger than the range in the launch conditions required to make the putt. The probability of the player making a putt will in these cases then be approximately proportional to the areas of the required launch conditions as given in the launch-speed – launch-angle space. Pelz [10] found that professional golfers make approximately 50% of putts from a distance of 6 ft. Using this value to scale the areas of required launch conditions, as given in launch-speed – launch-angle space, allows for the probability of making putts for other distances and other conditions to be determined. The result for a level green is shown in Fig. 11 with the probability of making a putt shown for hole distances ranging from 6 to 30 ft. Also shown is the range of success of professional golfers, as given by Pelz, in making putts at these same distances. As is seen, the general dependence of the probability of making a putt on hole distance, as predicted by the putting model, agrees well with the results of professional golfers.

4.2. Putting on uphill and downhill slopes

In the case of putting directly uphill or downhill, $\theta = 0°$, the acceleration of the golf ball can be determined by setting ϕ and β equal to 0° in (17b) and then substituting into (11b). The result is

$$a_y = -\frac{5}{7}g(\rho_g \cos\varphi + \sin\varphi) \tag{36}$$

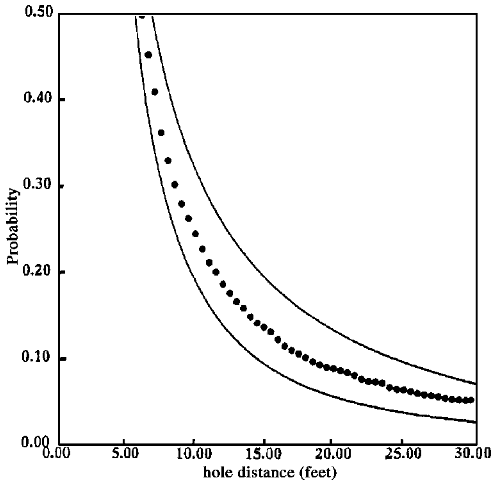


Fig. 11. The calculated probabilities of making a putt (dotted line) compared with the range of results of professional golfers as given by Pelz (continuous line).

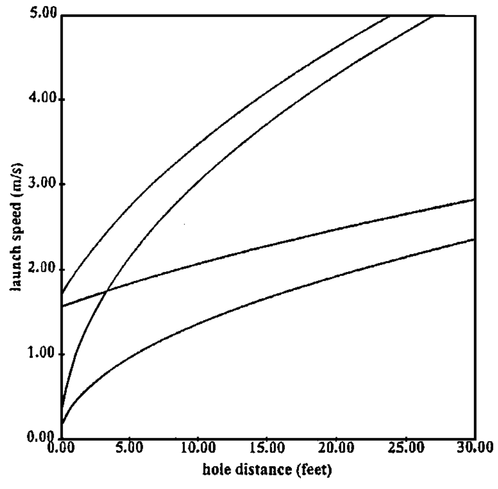


Fig. 12. The range of possible launch speeds for a successful putt as a function of hole distance in the case of a green with $\rho_g = 0.131$ and with an uphill slope of 5° (higher range) and a downhill slope of 5° (lower range).

The speed of the golf ball, v_f, when it reaches the hole, located a distance y away, will then be given by

$$v_f = (v_o^2 - \left(\frac{10}{7}\right)(\rho_g \cos\varphi + \sin\varphi)gy)^{1/2} \tag{37}$$

Therefore, in the case of direct impacts, where the range of impact speeds that result in the capture of the golf ball on an green, with an uphill slope of $\varphi = 5°$, is given by

$$0 < v_f < 1.71 \ \text{m/s} \tag{38}$$

the required launch speeds will be given by

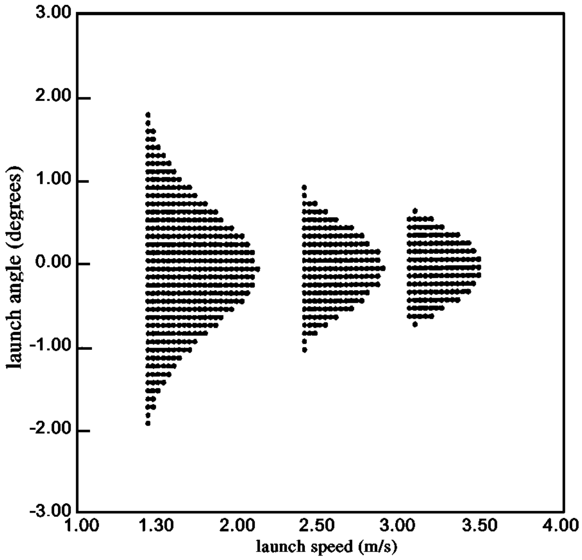
$$\left(\left(\frac{10}{7}\right)(\rho_g \cos\varphi + \sin\varphi)gy\right)^{1/2} < v_o < \left((1.71 \ \text{m/s})^2 - \left(\frac{10}{7}\right)(\rho_g \cos\varphi + \sin\varphi)gy\right)^{1/2} \tag{39}$$

Similarly in the case of downhill slopes

$$\left(\left(\frac{10}{7}\right)(\rho_g \cos\varphi + \sin\varphi)gy\right)^{1/2} < v_o < \left((1.56 \ \text{m/s})^2 - \left(\frac{10}{7}\right)(\rho_g \cos\varphi + \sin\varphi)gy\right)^{1/2} \tag{40}$$

where $\varphi = -5°$. Figure 12 shows the range of possible launch speeds for a successful putt as a function of hole distance in the case of greens with $\rho_g = 0.131$ and with uphill and downhill slopes of 5°. As is seen, the allowed range of launch speeds, for a given hole distance, is greater for downhill putts than it is for uphill putts. This is also true for the allowed range of launch angles, which is due to the convergence of misdirected putts towards the hole for downhill putts, as is shown in Fig. 3*c*, and the divergence of putts in the case of uphill putts, as is shown in Fig. 3*b*. The sets of launch conditions that resulted in successful putts in the case of a 10 ft putt, on a downhill slope of 5°, on a level green, and on a uphill slope of 5°, with a value for ρ_g of 0.131, are shown in Fig. 13. As previously mentioned the probability of a player making a putt would be expected to be approximately proportional to the areas

Fig. 13. The launch conditions that resulted in successful putts in the case of a 10 ft putt on a green with a downhill slope of 5° (left), on a level green (middle), and on a green with an uphill slope of 5° (right).

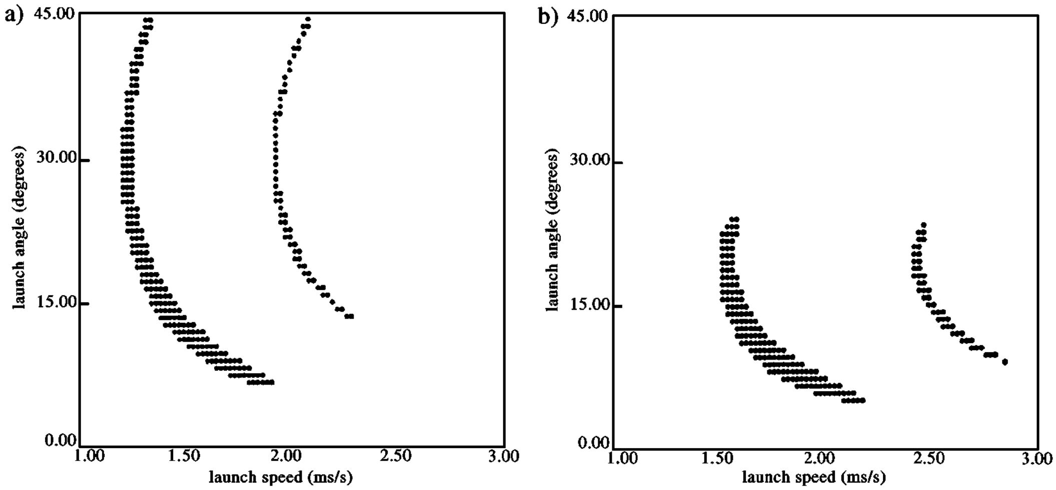


of the required launch conditions as given in launch-speed – launch-angle space. Therefore, as is seen in Fig. 13, the probability of making a downhill putt is significantly greater than the equivalent putt on a level green or an uphill putt. In the case shown, it is found that for a 10 ft putt the probability of making the downhill putt is approximately 2.9 times greater than the probability of making the uphill putt. This result would come as a surprise to most golfers and it needs to be pointed out that this would only be true for relatively small slopes where $a_{\mathrm{y}} < 0$. From (18), for an average green with $\rho_{\mathrm{g}} = 0.131$ the slope must be less than 7.5° while for a fast green with $\rho_{\mathrm{g}} = 0.065$ the slope must be less than 3.7°. Also, there is a trade off, for if the putt is missed the distance between the final stopping point of the golf ball and the hole is much greater for a downhill putt than for an uphill putt. For example, for the 10 ft putts of Fig. 13, if the golf ball is launched within the range of acceptable launch speeds for each of the given slopes but just misses the hole, it is found that in the case of the uphill putt the ball will end stopping as far as 2.9 ft from the hole while the downhill putts, if they just miss, will end up as far as 13.1 ft from the hole. The resulting increased probability of three putting along with the more delicate touch required for a downhill putt would lead most golfers still preferring an uphill putt.

4.3. Putts across the slope

Figure 14*a* shows the launch conditions required for 4 and 10 ft putts to a hole on the y-axis on a fast green with a value for ρ_{g} of 0.065 and with a slope of 5° in the x-direction. As can be seen, and also shown in Fig. 5*a*, there are, in general, two acceptable ranges of launch angles for each acceptable launch speed. Figure 14*b* shows the same putts in the case of an average green with a value of 0.131 for ρ_{g}. As is shown in this figure, and in Fig. 5*b*, for putts at the higher launch angles the golf ball will typically come to rest before it reaches the hole. As is indicated in both these figures, the greatest range in acceptable launch angles corresponds to approximately the minimum value of acceptable launch speeds. In the case of putts on average speed greens, such as is given in Fig 14*b*, this also corresponds to putts near the maximum allowed launch angle. This agrees with the advice given by Pelz [10], that based on his tests, putting towards the high side of the range of allowable launch angles provides the best statistical probability for making putts.

Fig. 14. The launch conditions that resulted in successful putts in the case of 4 ft (left) and 10 ft (right) putts on a green with a slope of 5° along the x-axis and with (a) $\rho_g = 0.065$, and (b) $\rho_g = 0.131$.



5. Conclusion

The dynamics and the resulting paths of the golf balls that have been presented provide a reasonable model for the motion of a golf ball on sloped greens. To further improve the model would require an investigation on the position of the contact area for a rolling ball on a sloped surface and the resulting contact forces and moments. The resulting required launch conditions that were determined from this model, along with Holmes' model, allowed for the determination of the dependence of the probability of making putts on the putt distance. The result agreed well with the actual performance of professional golfers. It was also found that the probability of making a downhill putt is much greater than the equivalent uphill putt. Although, for most golfers, the consequences of a missed putt would still lead to a preference for an uphill putt. Finally, it was found that in the case of putting across the slope of a green, the allowed variance in the launch angle is greatest when the launch speed is near its minimum allowed value.

The model presented in this paper could be applied, in general, to the topology of any green and it would be interesting to consider the variety of possibilities. Whether the results presented here would help a golfer improve their putting is debatable and, unfortunately, this author has not noticed any improvement in his game.

References

1. W.E. Lorensen and B. Yamrom. IEEE Comput. Graphics Appl. **12**, 35 (1992).
2. S.M. Alessandrini. SIAM Rev. **37**(3), 423 (1995).
3. B.W. Holmes. Am. J. Phys. **59**(2), 129 (1991).
4. A. Cochran and J. Stobbs. The search for the perfect swing. Lippincott, New York. 1968.
5. C.B. Daish. The physics of ball games. English Universities, London. 1972.
6. A. Domenech, T. Domenech, and J. Cebrian. Am. J. Phys. **55**(3), 231 (1987).
7. J. Witters and D. Duymelinck. Am. J. Phys. **54**(1), 80 (1986).
8. M. Hubbard and L.W. Alaways. *In* Proceedings of the 1998 World Scientific Congress of Golf. *Edited by* M.R. Farrally and A.J. Cochran. Human Kinetics, Leeds. 1999.
9. B.W. Holmes. Phys. Teach. **24**(7), 401 (1986).
10. D. Pelz. Dave Pelz's putting bible. Doubleday, New York. 2000.